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Contents

Preface

The uniformly distributed sequences have numerous applications in many branches of science as numerical methods, physics, financial mathematics, cryptography and others. This drives the significant scientific interest in these mathematical objects.

The thesis is organized as Preface, three chapters and References.

The first chapter has an auxiliary character. Here the definitions and properties of some classes of complete orthonormal function systems are presented. All considered function systems are constructed using Cantor number system. The definitions of the Vilenkin functions, the Haar functions and the $\mathcal{B}_s-$adic functions constructed in Cantor system are also reviewed. Additionally, here some elements of the theory of the uniformly distributed sequences are presented. Some quantitative measures for the irregularity of the distribution of sequences are also considered.

In the second chapter some applications of the functions of the system $\Gamma_{\mathcal{B}_s}$ to the theory of uniformly distributed sequences are shown. The notion of multidimensional modified integrals of the functions of the system $\Gamma_{\mathcal{B}_s}$ is introduced. Some classical results of the theory of the uniformly distributed sequences are presented in terms of the introduced modified integrals.

The third chapter considers some applications of the functions of the system $\Gamma_{\mathcal{B}_s}$ to the quasi-Monte Carlo integration in reproducing kernel Sobolev space. The notion of the mean square wort-case error of the integration, which is based on the arithmetic related with the functions of the system $\Gamma_{\mathcal{B}_s}$, is introduced. Some details related to this notion are presented. Two types of reproducing kernel weighted Sobolev spaces - unanchored and anchored are considered. The exact formulas for the mean square wort-case error of the integration in these spaces are proved. The notions of the weighted unanchored and the weighted anchored diaphony are introduced. The mean square wort-case errors of the integration in considered spaces are presented in terms of the corresponding versions of the diaphony.

Chapter 1

Some preliminary notations and statements

In this chapter of the thesis, we present the definitions and their main properties of the functions of some classes of complete orthonormal function systems constructed in Cantor systems.

Let $s \geq 1$ be a fixed integer, which will denote the dimension through the thesis.

1.1 Some notes about the Cantor number systems

The so-called Cantor systems are quite natural generalizations of the ordinary $b-$adic number system. The main components of the

$b-$adic number system are the base $b \geq 2$, which is a fixed integer, the set $\{0, 1, \ldots, b-1\}$ of the $b-$adic digits and the consecutive powers of b, thus the set $\{1 = b^0, b, b^2, \ldots\}$.

We can consider the Cantor systems as systems with variable bases. Thus, to the base b corresponds a sequence of bases

$$B = \{b_0, b_1, \ldots : b_i \geq 2 \text{ for } i \geq 0\}.$$

To the successive degrees of the base b corresponds the so-called generalized powers. Thus, we have the following correspondences: to $b^0 = 1$ corresponds $B_0 = 1$; to $b^1 = b$ corresponds $B_1 = b_0$; to $b^2 = b.b$ corresponds $B_2 = b_1.b_2$ and so one. In this way, to the sequence B corresponds the sequence $\{B_0, B_1, B_2, \ldots\}$ of the so-called *generalized powers*, defined in the following recursive manner:

$$B_0 = 1 \text{ and for } i \geq 0 \ B_{i+1} = B_i.b_i.$$

For the Cantor system with the above components we will use the notion $B-$adic number system.

An arbitrary integer number $k \geq 0$ has the unique $B-$adic representation of the form

$$k = \sum_{i=0}^{\nu} k_i B_i,$$

where for $0 \leq i \leq \nu$ we have that $k_i \in \{0, 1, \ldots, b_i - 1\}$ and $k_\nu \neq 0$.

An arbitrary real number $x \in [0, 1)$ has the $B-$adic representation of the form

$$x = \sum_{i=0}^{\infty} \frac{x_i}{B_{i+1}},$$

where for $i \geq 0$ we have that $x_i \in \{0, 1, \ldots, b_i - 1\}$. In the additional assumption, that for infinitely many indexes i $x_i \neq b_i - 1$ the above representation is also unique.

We will call that the above expressions are the $B-$adic representations of the integer k and the real x.

Now, we will give the concepts of the multidimensional Cantor systems. For this purpose, for $1 \leq j \leq s$ let

$$B_j = \{b_0^{(j)}, b_1^{(j)}, \ldots : b_i^{(j)} \geq 2 \text{ for } i \geq 0\}$$

be given sequences of bases and to denote $\mathcal{B}_s = (B_1, \ldots, B_s)$.

In every concrete case, we will use the multidimensional Cantor systems $\mathcal{B}_s$ to construct the corresponding function system.

1.2 Some classes of complete orthonormal function systems

In this paragraph, the constructions and the properties of some classes of complete orthonormal function systems are presented.

1.3 The function system $\Gamma_{\mathcal{B}_s}$

1.3.1 Definition and properties of the functions of the system $\Gamma_{\mathcal{B}_s}$

In 2021 Petrova [Pe 1] proposed the concept of a new function system constructed in $\mathcal{B}_s-$adic Cantor system, which is a naturale generalization of the system $\Gamma_{\mathbf{b}}$.

Definition 1.8 *For an arbitrary integer* $k \geq 0$ *and a real number* $x \in [0,1)$ *with the* B*-adic representations*

$$k = \sum_{i=0}^{\nu} k_i B_i \text{ and } x = \sum_{i=0}^{\infty} \frac{x_i}{B_{i+1}},$$

where for $i \geq 0$ $k_i,\ x_i \in \{0,1,\ldots,b_i-1\}$, $k_\nu \neq 0$ *and for infinitely many* i *we have* $x_i \neq b_i - 1$, *the* k*-th function* ${}_B\gamma_k : [0,1) \to \mathbb{C}$ *is defined as*

$${}_B\gamma_k(x) = e^{2\pi \mathbf{i}\left(\frac{k_0}{B_1}+\frac{k_1}{B_2}+\ldots+\frac{k_\nu}{B_{\nu+1}}\right)(x_0B_0+x_1B_1+\ldots+x_\nu B_\nu)}.$$

The set $\Gamma_B = \{{}_B\gamma_k(x) : k \in \mathbb{N}_0,\ x \in [0,1)\}$ is called $B-$adic function system.

Let the set $\mathcal{B}_s = (B_1, \ldots, B_s)$ be as above. We will present the concept of the multidimensional $\mathcal{B}_s-$adic functions.

Definition 1.9 *For an arbitrary vector* $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ *the* $\mathbf{k}-$*th function* ${}_{\mathcal{B}_s}\gamma_{\mathbf{k}} : [0,1)^s \to \mathbb{C}$ *is defined as*

$$
{}_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^{s} {}_{B_j}\gamma_{k_j}(x_j), \quad \mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s.
$$

The set $\Gamma_{\mathcal{B}_s} = \{{}_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x}) : \mathbf{k} \in \mathbb{N}_0^s,\ \mathbf{x} \in [0,1)^s\}$ is called $\mathcal{B}_s-$adic function system.

1.4 Uniformly distributed sequences

The object of the scientific discipline called *a theory of the uniformly distributed sequences* is the investigation of the distribution of the partial parts of the real numbers in the unit interval $[0,1)$. The ancestor of this discipline is the German mathematician Herman Weyl. In his important paper of 1916 *"Über die Gleichverteilung von Zahlen mod Eins"* he contemplated an attempt to specify the approximation theorem of Kronecker and the future development of the theory of the Diophantine approximations.

1.4.1 Definition of the notion uniformly distributed sequence

Let $\mathbf{a} = (a_1, \ldots, a_s)$ and $\mathbf{b} = (b_1, \ldots, b_s)$ be two vectors with real coordinates, i. e. $\mathbf{a}, \mathbf{b} \in \mathbb{R}^s$. We will call that $\mathbf{a} < \mathbf{b}$ if for each $1 \leq j \leq s$ the inequalities $a_j < b_j$ hold. The set of points $\mathbf{x} \in \mathbb{R}^s$ such that $\mathbf{a} \leq \mathbf{x} < \mathbf{b}$ we will denote by $[\mathbf{a}, \mathbf{b})$.

For an arbitrary vector $\mathbf{x} = (x_1, \ldots, x_s) \in \mathbb{R}$ its integer part will be $[\mathbf{x}] = ([x_1], \ldots, [x_s])$ and the fractional part of $\mathbf{x}$ is $\{\mathbf{x}\} = (\{x_1\}, \ldots, \{x_s\})$.

Let $\xi = (\mathbf{x}_n)_{n \geq 0}$ be an arbitrary sequence of vectors in $\mathbb{R}^s$. Let $J = \prod_{j=1}^{s} [a_j, b_j)$, where for $1 \leq j \leq s$ $0 \leq a_j < b_j \leq 1$, be an arbitrary subinterval of $[0,1)^s$ with a volume $\mu(J) = \prod_{j=1}^{s} (b_j - a_j)$. For an arbitrary positive integer number N let as above

$$A(J; N; \xi) = \#\{\mathbf{x}_n : 0 \leq n \leq N-1,\ \{\mathbf{x}_n\} \in J\}.$$

In the following definition the concept of uniformly distributed sequence is presented:

Definition 1.11 *The sequence $\xi = (\mathbf{x}_n)_{n \geq 0}$ of vectors is called* *uniformly distributed modulus 1 in* $\mathbb{R}^s$ *if the limit equality*

$$\lim_{N\to\infty} \frac{A(J;N;\xi)}{N} = \mu(J)$$

holds for each subinterval J *of* $[0,1)^s$.

1.4.2 Integral Weyl's criterion

In this section the main important result of the theory of the uniformly distributed sequences is the so-called *integral Weyl's criterion* is exposed.

1.4.3 Exponential Weyl's criterion

In this section the so-called *exponential Weyl's criterion*, is presented.

1.4.4 Some quantitative characteristics for the irregularity of the distributions of sequences

In Definitions 1.11 the concept of uniformly distributed sequence was presented. In fact, this definition presents the idea for the ideal uniform distribution. But in the analytical nature of the construction of each sequence an insurmountable irregularity is staked. In this sense, each sequence has own concrete distribution, which distinguishes from the ideal uniform distribution.

In the theory of the uniformly distributed sequences, the degree of deviation of the distribution of a concrete sequence from the ideal distribution is measured with special quantitative measures. In practice, they give quantitative estimation of inevitable quality called *irregularity of the distribution.* In general, these quantitative measures are different types of the discrepancy and the diaphony.

Let $\xi = (\mathbf{x}_n)_{n\geq 0}$ be an arbitrary sequence of points in $\mathbb{R}^s$. For an arbitrary subinterval J of $[0,1)^s$ and each integer $N \geq 1$ the quantity

$$R(J;N;\xi) = \frac{A(J;N;\xi)}{N} - \mu_s(J),$$

where $\mu_s(J)$ is the Lebesgue measure of the interval J, is called *local discrepancy* of the sequence ξ. By taking different norms of the local discrepancy, for example the L_p-norm, $1 \leq p < \infty$, the $L_\infty-$norm, we obtain different kinds of the discrepancy.

Let $\mathcal{J}$ and $\mathcal{J}^*$ denote the sets of subintervals of $[0,1)^s$ respectively of the form $J = \prod_{j=1}^{s}[u_j, v_j)$ and $J^* = \prod_{j=1}^{s}[0, v_j)$, where for $1 \leq j \leq s$ $0 \leq u_j < v_j \leq 1$. For an arbitrary vector $\mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s$ let us denote $[\mathbf{0}, \mathbf{x}) = [0, x_1) \times \ldots \times [0, x_s)$. We will give the following definitions:

Definition 1.12 *For each integer number $N \geq 1$ the extreme discrepancy $D_N(\xi)$, the star-discrepancy $D_N^*(\xi)$ and the quadratic discrepancy $T_N(\xi)$ of the first N elements of the sequence ξ are defined respectively as*

$$D_N(\xi) = \sup_{J \in \mathcal{J}} \left| \frac{A(J;N;\xi)}{N} - \mu_s(J) \right|,$$

$$D_N^*(\xi) = \sup_{J^* \in \mathcal{J}^*} \left| \frac{A(J^*;N;\xi)}{N} - \mu_s(J^*) \right|$$

and

$$T_N(\xi) = \left(\int_{[0,1]^s} \left| \frac{A([\mathbf{0},\mathbf{x});N;\xi)}{N} - x_1 \dots x_s \right|^2 dx_1 \dots dx_s \right)^{\frac{1}{2}}.$$

Well known is the following result:

Theorem 1.1 *The sequence ξ is uniformly distributed modulus 1 in $\mathbb{R}^s$ if and only if one of the following equivalent limit equalities hold*

$$\lim_{N\to\infty} D_N(\xi) = 0, \ \lim_{N\to\infty} D_N^*(\xi) = 0 \ u \ \lim_{N\to\infty} T_N(\xi) = 0.$$

Chapter 2

The $\mathcal{B}_s-$adic functions and uniform distribution of sequences

2.1 Notices connected with some classical quantitative results of the theory of the uniformly distributed sequences

In this section many explanations about some classical results of the quantitative theory of the uniformly distributed sequences are presented.

2.2 Multidimensional modified integrals of the functions of the system $\Gamma_{\mathcal{B}_s}$

Following Fine [Fi 1] for an arbitrary integer $k \geq 0$ and a real $x \in [0, 1)$ we will again consider the integral of the function ${}_B\gamma_k(x)$, thus let

$$ {}_B\Phi_k(x) = \int_0^x {}_B\gamma_k(t)dt. $$

To introduce the concept of the multidimensional modified integrals of the functions of the system $\Gamma_{\mathcal{B}_s}$ we need some preliminary explanations. Let us denote $S = \{1, 2, \ldots, s\}$. For an arbitrary integer $0 \leq u \leq s$ let $A_u = \{\alpha_1, \ldots, \alpha_u : 1 \leq \alpha_1 < \ldots < \alpha_u \leq s\}$ be an arbitrary subset of S. Obviously we have C_s^u choices of the subsets A_u.

Let $C_{s-u} = S \setminus A_u$ and to denote $C_{s-u} = \{\beta_1, \ldots, \beta_{s-u} : 1 \leq \beta_1 < \ldots < \beta_{s-u} \leq s\}$.

In the case when u=0 we will assume that $A_0 = \emptyset$ and $C_s = S$. When $u = s$ we have that $A_s = S$ and $C_0 = \emptyset$.

Let us assume that $u = 0$. To the value $u = 0$ corresponds the s-dimensional vector $\mathbf{k} = \mathbf{0}$. Let us introduce the notion of *modified* *integral* of the function ${}_{\mathcal{B}_s}\gamma_{\mathbf{0}}(\mathbf{x})$ as

$$ {}_{\mathcal{B}_s}\Phi_{\mathrm{mod},\mathbf{0}}(\mathbf{x}) = \prod_{j=1}^{s}(1-x_j) - \frac{1}{2^s}, \ \mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s. $$

Now, let us assume that $1 \leq u \leq s$. Let $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ be an arbitrary vector and let us assume that exactly u in number coordinates of $\mathbf{k}$ are different than zero and these are $k_{\alpha_1}, \ldots, k_{\alpha_u}$ and $s-u$ coordinates of $\mathbf{k}$ are equal to zero and these are $k_{\beta_1}, \ldots, k_{\beta_{s-u}}$. Let us assume that the vector $\mathbf{k}$ satisfies the following condition:

(C) For $1 \leq j \leq u$ the coordinate k_{α_j} to present of the form

$$ k_{\alpha_j} = k_{g_j}^{(\alpha_j)}.B_{g_j}^{(\alpha_j)} + k'_{\alpha_j}, $$

where $g_j \geq 0$, $0 \leq k'_{\alpha_j} \leq B_{g_j}^{(\alpha_j)} - 1$ and $k_{g_j}^{(\alpha_j)} \in \{1, \ldots, b_{g_j}^{(\alpha_j)} - 1\}$.

Let $\mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s$ be an arbitrary vector. For $1 \leq u \leq s$ and the set C_{s-u} let us denote $\mathbf{x}^{(s-u)} = (x_{\beta_1}, \ldots, x_{\beta_{s-u}})$ and define the function

$$ \Omega(\mathbf{x}^{(s-u)}) = \begin{cases} \prod_{j=1}^{s-u}(1-x_{\beta_j}), & \text{if} \quad 1 \leq u \leq s-1, \\ \quad 1, & \text{if} \qquad u = s. \end{cases} $$

We will introduce the notion of *modified integral of rang* u of the function ${}_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x})$ as

$$ {}_{\mathcal{B}_s}\Phi_{(g_1,\ldots,g_u),(k_{g_1}^{(\alpha_1)},\ldots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\ldots,k_{\alpha_u})}(\mathbf{x}) $$

$$= (-1)^u \prod_{j=1}^{u} {}_{B_{\alpha_j}}\Phi_{k_{\alpha_j}}(x_{\alpha_j}) \times \Omega(\mathbf{x}^{(s-u)})$$

$$-\prod_{j=1}^{u} \left[\frac{1}{\left[B_{g_j+1}^{(\alpha_j)}\right]^2} . \frac{e^{2\pi \mathbf{i} \frac{k_{g_j}^{(\alpha_j)}}{b_{g_j}^{(\alpha_j)}}} - 1}{e^{2\pi \mathbf{i}\left(\frac{k_0^{(\alpha_j)}}{b_0^{(\alpha_j)}} + \ldots + \frac{k_{g_j}^{(\alpha_j)}}{b_0^{(\alpha_j)} \ldots b_{g_j}^{(\alpha_j)}}\right)} - 1} . \Delta(b_{g_j}^{(\alpha_j)}; k_{g_j}^{(\alpha_j)}) \right] \frac{1}{2^{s-u}},$$

$$\mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s.$$

2.3 Preliminary results

In this paragraph, many useful results are proved.

2.4 Applications of the modified integrals of the functions of the system $\Gamma_{\mathcal{B}_s}$ to the theory of the uniformly distributed sequences

2.4.1 The LeVeque's inequality

Theorem 2.1 (The LeVeque's inequality)

Let $\xi_N = \{x_0, \ldots, x_{N-1}\}$ be an arbitrary net composed by N points in $[0,1)$. Then, the extreme discrepancy $D(\xi_N)$ of the net ξ_N satisfies *the inequality*

$$D^3(\xi_N) \leq 12 \sum_{g=0}^{\infty} \sum_{k_g=1}^{b_g-1} \sum_{k=k_g B_g}^{(k_g+1)B_g-1} \left| \frac{1}{N} \sum_{k=0}^{N-1} {}_B\Phi_{g,k_g,k}(x_n) \right|^2 .$$

2.4.2 The Koksma's formula

Theorem 2.2 *(***The Koksma's formula***) Let* $\xi_N = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$ *be an arbitrary net composed by* $N \geq 1$ *points in* $[0,1)^s$*. Then, the quadratic discrepancy* $T(\xi_N)$ *of the net* ξ_N *satisfies the equality*

$$T^2(\xi_N) = \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_{\mathcal{B}_s}\Phi_{\mathrm{mod},\mathbf{0}}(\mathbf{x}_n) \right|^2$$

$$+ \sum_{u=1}^{s} \sum_{\{\alpha_1,\ldots,\alpha_u\}\subseteq S} \prod_{j=1}^{u} \sum_{g_j=0}^{\infty} \sum_{k_{g_j}^{(\alpha_j)}=1}^{b_{g_j}^{(\alpha_j)}-1} \sum_{k_{\alpha_j}=k_{g_j}^{(\alpha_j)}.B_{g_j}^{(\alpha_j)}}^{(k_{g_j}^{(\alpha_j)}+1).B_{g_j}^{(\alpha_j)}-1} \times$$

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} {}_{\mathcal{B}_s}\Phi_{(g_1,\ldots,g_u),(k_{g_1}^{(\alpha_1)},\ldots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\ldots,k_{\alpha_u})}(\mathbf{x}_n) \right|^2 .$$

2.4.3 The Erdös-Turán-Koksma's inequality

Theorem 2.3 *(***The Erdös-Turán-Koksma's inequality***) Let us assume that the coordinate sequences of the* $\mathcal{B}_s-$*adic system are* *bounded from above, i. e., there exists an absolute constant C such that for $1 \leq j \leq s$ and each $i \geq 0$ the inequality $b_i^{(j)} \leq C$ holds. Let us denote $b = \min\limits_{1\leq j\leq s} \min\limits_{i\geq 0} b_i^{(j)}$. Let us define the constant*

$$K(C;s) = \begin{cases} \dfrac{9}{4}, & \text{if} \quad C = 2, \\ \left[1 + \left(\dfrac{C^2}{8}\right)^s\right]^2, & \text{if} \quad C \geq 3. \end{cases}$$

Let $\xi_N = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$ be an arbitrary net composed by N points in $[0,1)^s$. Then, for an arbitrary integer $M > 1$ the inequality holds

$$T^2(\xi_N) \leq K(C;s)\left[1 + (C-1)\frac{b}{b-1}\right]^s \cdot \frac{1}{b^M} + \left(\frac{1}{N}\sum_{n=0}^{N-1} {}_{\mathcal{B}_s}\Phi_{\mathrm{mod},\mathbf{0}}(\mathbf{x}_n)\right)^2$$

$$+ \sum_{u=1}^{s} \sum_{\{\alpha_1,\ldots,\alpha_u\}\subseteq S} \prod_{j=1}^{u} \sum_{g_j=0}^{M-1} \sum_{k_{g_j}^{(\alpha_j)}=1}^{b_{g_j}^{(\alpha_j)}-1} \sum_{k_{\alpha_j}=k_{g_j}^{(\alpha_j)}.B_{g_j}^{(\alpha_j)}}^{(k_{g_j}^{(\alpha_j)}+1).B_{g_j}^{(\alpha_j)}-1} \times$$

$$\left|\frac{1}{N}\sum_{n=0}^{N-1} {}_{\mathcal{B}_s}\Phi_{(g_1,\ldots,g_u),(k_{g_1}^{(\alpha_1)},\ldots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\ldots,k_{\alpha_u})}(\mathbf{x}_n)\right|^2.$$

2.4.4 The integral Weyl's criterion

Theorem 2.4 *(***The integral Weyl's criterion***) Let $\xi = (\mathbf{x}_n)_{n\geq 0}$ be an arbitrary sequence of points in $[0,1)^s$. The sequence ξ is uniformly distributed in $[0,1)^s$ if and only if the following conditions hold:*

(i) The limit equality

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}{}_{\mathcal{B}_s}\Phi_{\mathrm{mod},\mathbf{0}}(\mathbf{x}_n)=0$$

holds;

(ii) For each choice of the parameters $1\le u\le s$, $\{\alpha_1,\ldots,\alpha_u\}\subseteq S$, $(g_1,\ldots,g_u)\in\mathbb{N}_0^s$, *for* $1\le j\le u$ $k_{g_j}^{(\alpha_j)}\in\{1,\ldots,b_{g_j}^{(\alpha_j)}-1\}$ *and* $k_{g_j}^{(\alpha_j)}.B_{g_j}^{(\alpha_j)}\le k_{\alpha_j}\le(k_{g_j}^{(\alpha_j)}+1).B_{g_j}^{(\alpha_j)}-1$ *the limit equality holds*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}{}_{\mathcal{B}_s}\Phi_{(g_1,\ldots,g_u),(k_{g_1}^{(\alpha_1)},\ldots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\ldots,k_{\alpha_u})}(\mathbf{x}_n)=0.$$

2.5 Concluding remarks

In this section, some remarks of general character for the applications of the functions of the system $\Gamma_{\mathcal{B}_s}$ to the theory of the uniformly distributed sequences are developed.

Chapter 3

Multivariate integration in weighted Sobolev spaces

In this chapter of the thesis, some problems of the Quasi-Monte Carlo integration in weighted Sobolev spaces are considered.

3.1 General remarks about quasi-Monte Carlo integration in reproducing kernel Hilbert spaces

Following Aronszajan [Ar 1], we will remind the notion of a reproducing kernel for a Hilbert space.

Definition 3.1 *Let F be a class of functions defined on E forming Hilbert space. The function $K(x, y)$ of $x, y \in E$ is called a reproducing* *kernel for the space F if the following properties hold:*

1.) For every fixed $y \in E$ the kernel $K(x, y)$, as a function of x, belongs to F;

*2.) (**reproducing property**) For every function $f \in F$ and every $y \in E$ the equality*

$$f(y) =< f(x), K(x, y) >_x$$

holds. Here, the subscript x indicates that the inner product is given with respect to the variable x.

The inner product $< \cdot, \cdot >$ in the space F generates a norm $||f||$, thus we have that $||f|| = \sqrt{< f(x), f(x) >}$. Usually, the space F is considered as a set of functions with finite norm, thus $F = \{f(x),\ x \in E : ||f|| < +\infty\}$.

The multidimensional Hilbert spaces $H_{s,\gamma}$ are defined as a tensor product of the corresponding one dimensional Hilbert spaces H_{1,γ_j}, thus we have that $H_{s,\gamma} = H_{1,\gamma_1} \otimes \ldots \otimes H_{1,\gamma_s}$.

For $1 \leq j \leq s$ let $K_{1,\gamma_j}(x_j, y_j)$, where $x_j, y_j \in [0, 1)$, be the reproducing kernel of the one dimensional Hilbert space H_{1,γ_j}. Then, the reproducing kernel of the multidimensional Hilbert space $H_{s,\gamma}$ is defined as

$$K_{s,\gamma}(\mathbf{x},\mathbf{y}) = \prod_{j=1}^{s} K_{1,\gamma_j}(x_j, y_j),$$

$$\mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s, \ \mathbf{y} = (y_1, \ldots, y_s) \in [0,1)^s.$$

The technique of the numerical integration in reproducing kernel Hilbert spaces is given as follows: Let $H_s(K)$ be a Hilbert space generated by the reproducing kernel K with an inner product $< \cdot >_{H_s(K)}$, which engenders the norm $|| \cdot ||_{H_s(K)}$.

We will consider the integral

$$I_s(f) = \int_{[0,1]^s} f(\mathbf{x}) d\mathbf{x}, \ f \in H_s(K).$$

The integral $I_s(f)$ is approximated by a Quasi-Monte Carlo algorithm

$$Q_s(f; P_N) = \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n),$$

where $P_N = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$ is a deterministic sample point net in $[0,1)^s$.

In the next definition, the main notion of the theory of the Quasi-Monte Carlo integration in reproducing kernel Hilbert spaces $-$ the notion of worst-case error of the integration is presented.

Definition 3.2 *The worst-case error of the integration in the space $H_s(K)$ by using the net P_N of nodes is defined as*

$$e(H_s(K); P_N) = \sup_{f \in H_s(K),\ ||f||_{H_s(K)} \leq 1} |I_s(f) - Q_s(f; P_N)|.$$

3.2 Some notes about the multivariate integration in weighted Sobolev spaces

In this paragraph, some general notes about the analytical structure of the weighted Sobolev spaces are developed.

3.2.1 A mean square worst-case error of the integration in weighted Sobolev spaces

Let $P_N = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$ be an arbitrary net of N points in $[0,1)^s$. Let $\sigma \in [0,1)^s$ be an arbitrary and fixed vector. By using the vector $\oplus_{\mathcal{B}_s}^{[0,1)^s}$ let us define the net $P_N(\sigma) = \{\mathbf{x}_0 \oplus_{\mathcal{B}_s}^{[0,1)^s} \sigma, \ldots, \mathbf{x}_{N-1} \oplus_{\mathcal{B}_s}^{[0,1)^s} \sigma\}$, which we will call a *digitally $\mathcal{B}_s-$adic shifted net.*

The following Definitions are presented:

Definition 3.3 *The mean square worst-case error of the integra-*

tion in the space $H_s(K)$ by using the net P_N is defined as

$$\tilde{e}^2(H_s(K); P_N) = \int_{[0,1]^s} e^2(H_s(K); P_N(\sigma))d\sigma.$$

Definition 3.4 *For an arbitrary reproducing kernel K we define the associated digitally $\mathcal{B}_s-$adic shifted kernel as*

$$K_{ds}(\mathbf{x}, \mathbf{y}) = \int_{[0,1]^s} K(\mathbf{x} \oplus_{\mathcal{B}_s}^{[0,1)^s} \sigma, \mathbf{y} \oplus_{\mathcal{B}_s}^{[0,1)^s} \sigma)d\sigma,\ \mathbf{x}, \mathbf{y} \in [0,1)^s.$$

The following theorem is proved:

Theorem 3.1 *Let $H_s(K)$ be an arbitrary Hilbert space generated by the reproducing kernel K. Let P_N be an arbitrary net composed by N points in $[0,1)^s$. Then, the mean square worst-case error of the integration in the space $H_s(K)$ by using the net P_N satisfies the equality*

$$\tilde{e}(H_s(K); P_N) = e(H_s(K_{ds}); P_N),$$

i. e. the mean square worst-case error of the integration in the space $H_s(K)$ by using the net P_N is equal to the ordinary worst-case error of the integration in the Hilbert space $H_s(K_{ds})$ generated by the associated $\mathcal{B}_s-$adic digitally shifted kernel K_{ds} and by using the same net P_N.

3.3 Some preliminary results

In this paragraph, some preliminary results, which are essentially used in the next paragraphs of the thesis are present.

3.4 General remarks about reproducing kernel Sobolev spaces

In this paragraph, many details about structure and the reproducing kernels of the unanchored and anchored weights Sobolev spaces are presented.

3.5 Multidimensional integration in the unanchored weighted Sobolev space $H_{Sob,s,\gamma}$

In this paragraph, we will consider problems related to the multidimensional integration in the unanchored weighted Sobolev space $H_{Sob,s,\gamma}$.

3.5.1 A construction of the space $H_{Sob,s,\gamma}$

Following Sloan and Woźniakowski [SlWo 1], for an arbitrary vector γ of positive weights, we will consider the weighted unanchored Sobolev space $H_{Sob,s,\gamma}$. To present the inner product in the space $H_{Sob,s,\gamma}$, we need some notations: For an arbitrary vector $\mathbf{x} \in [0,1)^s$ and a subset $u \subseteq \{1,2,\ldots,s\}$ let the symbol $\mathbf{x}_u$ denote the vector from $[0,1]^{|u|}$ consisting of the components of $\mathbf{x}$ with indexes in u and $\mathbf{x}_{-u}$ – the vector of components of $\mathbf{x}$ with indexes not in u. Also we denote $\gamma_u = \prod_{j\in u} \gamma_j$ and $d\mathbf{x}_u = \prod_{j\in u} dx_j$.

For two functions f and g, their inner product is defined as

$$\langle f,g\rangle_{s,\gamma} = \sum_{u\subseteq\{1,\ldots,s\}} \gamma_u^{-1} \int_{[0,1]^{|u|}} \left[\int_{[0,1]^{s-|u|}} \frac{\partial^{|u|}}{\partial \mathbf{x}_u} f(\mathbf{x}) d\mathbf{x}_{-u} \times \right.$$

$$\left. \int_{[0,1]^{s-|u|}} \frac{\partial^{|u|}}{\partial \mathbf{x}_u} g(\mathbf{x}) d\mathbf{x}_{-u}\right] d\mathbf{x}_u.$$

In this way, the norm $||f||_{s,\gamma}$ is given by $||f||_{s,\gamma} = \sqrt{\langle f,f\rangle_{s,\gamma}}$. Finally, the space $H_{Sob,s,\gamma}$ is defined as $H_{Sob,s,\gamma} = \{f : ||f||_{s,\gamma} < +\infty\}$.

For an arbitrary real $\gamma > 0$ the function

$$K_{1,\gamma}(x,y) = 1 + \gamma \left[\frac{1}{2} B_2(|x-y|) + B_1(x)B_1(y)\right], \; x,y \in [0,1).$$

is introduced and the following Lemma is proved.

Lemma 3.4 *The function $K_{s,\gamma}$ is the reproducing kernel of the Hilbert space $H_{Sob,s,\gamma}$.*

3.6 A formula for the mean square worst-case error of the integration in the space $H_{Sob,s,\gamma}$

In the next theorem, we present the formula for the mean square worst-case error of the integration in the space $H_{Sob,s,\gamma}$.

Theorem 3.2 *For an arbitrary vector $\gamma = (\gamma_1, \ldots, \gamma_s)$ of coordinate weights and a vector $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ let the coefficient $\widehat{K}_{s,\gamma}(\mathbf{k}, \mathbf{k})$ be defined in the condition of Lemma 3.5.*

Let $P_N = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$ be an arbitrary net composed by N points in $[0, 1)^s$.

Then, the mean square worst-case error of the integration in the space $H_{Sob,s,\gamma}$ by using the net P_N satisfies the equality

$$\tilde{e}^2 \left(H_{Sob,s,\gamma}; P_N\right) = -1 + \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} K_{ds,s,\gamma}(\mathbf{x}_n, \mathbf{x}_m).$$

Also, the equality holds

$$\tilde{e}^2 \left(H_{Sob,s,\gamma}; P_N\right)$$

$$= \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \sum_{\mathbf{k} \in \mathbb{N}_0^s \setminus \{\mathbf{0}\}} \widehat{K}_{s,\gamma}(\mathbf{k}, \mathbf{k})_{\mathcal{B}_s} \gamma_{\mathbf{k}}(\mathbf{x}_n)_{\mathcal{B}_s} \overline{\gamma}_{\mathbf{k}}(\mathbf{x}_m).$$

3.7 A notion of the unanchored weighted $(\Gamma_{\mathcal{B}_s};\gamma)-$ diaphony

In the next definition, we introduce the concept of the so-called unanchored weighted $(\Gamma_{\mathcal{B}_s};\gamma)-$diaphony.

Definition 3.5 *For an arbitrary integer* $N \geq 1$ *the unanchored weighted* $(\Gamma_{\mathcal{B}_s};\gamma)-$*diaphony of the first* N *elements of the sequence* $\xi = (\mathbf{x}_n)_{n\geq 0}$ *of points in* $[0,1)^s$ *is defined as*

$$F_N(\Gamma_{\mathcal{B}_s};\gamma;\xi) = \left(\frac{1}{C(\gamma;\mathcal{B}_s)} \sum_{\mathbf{k}\in\mathbb{N}_0^s\setminus\{\mathbf{0}\}} \rho(\gamma;\mathcal{B}_s;\mathbf{k}) \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x}_n) \right|^2 \right)^{\frac{1}{2}},$$

where the coefficient $\rho(\gamma;\mathcal{B}_s;\mathbf{k})$ *and the constant* $C(\gamma;\mathcal{B}_s)$ *are defined respectively by the equalities (3.46) and (3.49).*

In the next theorem, we show that the unanchored weighted $(\Gamma_{\mathcal{B}_s};\gamma)-$diaphony is quantitative measure for the irregularity of the distribution of sequences. So, the following theorem holds:

Theorem 3.3 *The sequence* ξ *is uniformly distributed in* $[0,1)^s$ *if and only if the limit equality*

$$\lim_{N\to\infty} F_N(\Gamma_{\mathcal{B}_s};\gamma;\xi) = 0$$

holds for each choice of the vector γ *of coordinate weights.*

3.8 A relationship between the mean square worst-case error and the unanchored weighted $(\Gamma_{\mathcal{B}_s};\gamma)-$diaphony

Theorem 3.4 *Let P_N be an arbitrary net by N points in $[0,1)^s$. Then, the mean square worst-case error $\tilde{e}(H_{Sob,s,\gamma};P_N)$ of the integration in the space $H_{Sob,s,\gamma}$ by using the net P_N of nodes and the unanchored weighted $F(\Gamma_{\mathcal{B}_s};\gamma)$ of this net are related with the equality*

$$\tilde{e}(H_{Sob,s,\gamma};P_N) = \sqrt{C(\gamma;\mathcal{B}_s)}.F(\Gamma_{\mathcal{B}_s};\gamma;P_N),$$

where the constant $C(\gamma;\mathcal{B}_s)$ is defined by the equality (3.49).

3.9 Multidimensional integration in the anchored weighted Sobolev space $H_{Sob,s,\gamma,\mathbf{w}}$

3.9.1 A construction of the space $H_{Sob,s,\gamma,\mathbf{w}}$

In this section, we consider the anchored case of a choice of the reproducing kernel. Following Hickernell [Hi 2] and Novak and Woźniakowski [NoWo 1], we briefly will recall the details:

Let $\mathbf{w} = (w_1,\ldots,w_s) \in [0,1]^s$ be a fixed vector, which we will call "*anchor*".

To present the inner product in the space $H_{Sob,s,\gamma,\mathbf{w}}$, we need some notations. For an arbitrary subset $u \subseteq \{1,\dots,s\}$ let $|u|$ denote the cardinality of the set u. For a vector $\mathbf{x} \in [0,1)^s$ the symbol $\mathbf{x}_u$ denotes the vector from $[0,1)^{|u|}$ consisting the components of $\mathbf{x}$ with indexes in u. By using the anchor vector $\mathbf{w}$ the notation $(\mathbf{x}_u, \mathbf{w}_{-u})$ will denote the vector from $[0,1]^s$ with coordinate x_j when $j \in u$ and w_j when $j \in \{1,\dots,s\} \setminus u$.

Also, we denote $\gamma_u = \prod_{j \in u} \gamma_j$, $d\mathbf{x}_u = \prod_{j\in u} dx_j$.

For two functions f and g their inner product is defined as

$$\langle f, g\rangle_{s,\gamma,\mathbf{w}}$$

$$= \sum_{u\subseteq\{1,\dots,s\}} \gamma_u^{-1} \int_{[0,1]^{|u|}} \frac{\partial^{|u|}}{\partial \mathbf{x}_u} f(\mathbf{x}_u, \mathbf{w}_{-u}) \frac{\partial^{|u|}}{\partial \mathbf{x}_u} g(\mathbf{x}_u, \mathbf{w}_{-u}) d\mathbf{x}_u.$$

In this way, the norm $||f||_{s,\gamma,\mathbf{w}}$ is given by $||f||_{s,\gamma,\mathbf{w}} = \sqrt{\langle f, g\rangle_{s,\gamma,\mathbf{w}}}$.

Finally, the space $H_{Sob,s,\gamma,\mathbf{w}}$ is defined as $H_{Sob,s,\gamma,\mathbf{w}} = \{f : ||f||_{s,\gamma,\mathbf{w}} < +\infty\}$.

For arbitrary reals $\gamma > 0$ and $w \in [0,1]$ we will consider the function

$$K_{1,\gamma,w}(x,y) = 1 + \gamma.\mu_w(x,y),\ x,y \in [0,1),$$

where the quantity $\mu_w(x,y)$ is defined as

$$\mu_w(x,y) = \frac{|x-w| + |y-w| - |x-y|}{2}.$$

For an arbitrary vector of coordinates $\gamma = (\gamma_1, \dots, \gamma_s)$ and the given anchor vector $\mathbf{w} = (w_1, \dots, w_s)$ the multidimensional function $K_{s,\gamma,\mathbf{w}}(\mathbf{x}, \mathbf{y})$ is defined as

$$K_{s,\gamma,\mathbf{w}}(\mathbf{x}, \mathbf{y}) = \prod_{j=1}^{s} K_{1,\gamma_j,w_j}(x_j, y_j),$$

$$\mathbf{x} = (x_1, \dots, x_s) \in [0,1)^s,\ \mathbf{y} = (y_1, \dots, y_s) \in [0,1)^s.$$

The following lemma is proved:

Lemma 3.1 *The function $K_{s,\gamma,\mathbf{w}}$ is the reproducing kernel of the Hilbert space $H_{Sob,s,\gamma,\mathbf{w}}$.*

3.10 A formula for the mean square worst-case error of the integration in the space $H_{Sob,s,\gamma,\mathbf{w}}$

In the next theorem, we present the formula for the mean square worst-case error of the integration in the space $H_{Sob,s,\gamma,\mathbf{w}}$.

Theorem 3.5 *For an arbitrary vector* $\gamma = (\gamma_1, \ldots, \gamma_s)$ *of coordinate weights, a given anchor vector* $\mathbf{w} = (w_1, \ldots, w_s) \in [0,1]^s$ *and an arbitrary vector* $\mathbf{k} \in \mathbb{N}_0^s$ *let the coefficient* $\widehat{K}_{s,\gamma,\mathbf{w}}(\mathbf{k},\mathbf{k})$ *be defined in the condition of Lemma 3.7.*

Let $P_N = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$ *be an arbitrary net composed by* N *points in* $[0,1)^s$.

Then, the mean square worst-case error of the integration in the space $H_{Sob,s,\gamma,\mathbf{w}}$, *by using the net* P_N, *satisfies the equality*

$$\tilde{e}^2\left(H_{Sob,s,\gamma,\mathbf{w}}; P_N\right) = -\prod_{j=1}^{s}\left[1 + \gamma_j\left(w_j^2 - w_j + \frac{1}{3}\right)\right]$$

$$+\frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} K_{ds,s,\gamma,\mathbf{w}}(\mathbf{x}_n, \mathbf{x}_m).$$

Also, the equality holds

$$\tilde{e}^2\left(H_{Sob,s,\gamma,\mathbf{w}}; P_N\right)$$

$$= \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\sum_{\mathbf{k}\in\mathbb{N}_0^s\setminus\{\mathbf{0}\}} \widehat{K}_{s,\gamma,\mathbf{w}}(\mathbf{k},\mathbf{k})_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x}_n)_{\mathcal{B}_s}\overline{\gamma}_{\mathbf{k}}(\mathbf{x}_m).$$

3.11 A notion of the anchored weighted $(\Gamma_{\mathcal{B}_s}; \gamma; \mathbf{w})-$ diaphony

In this paragraph, we introduce the notion of an anchored weighted

$(\Gamma_{\mathcal{B}_s};\gamma;\mathbf{w})-$diaphony.

Definition 3.6 *For an arbitrary integer $N \geq 1$ the anchored weighted $(\Gamma_{\mathcal{B}_s};\gamma;\mathbf{w})-$diaphony of the first N elements of the sequence $\xi = (\mathbf{x}_n)_{n\geq 0}$ of points in $[0,1)^s$ is defined as*

$$F_N(\Gamma_{\mathcal{B}_s};\gamma;\mathbf{w};\xi)$$

$$= \left(\frac{1}{C(\gamma;\mathbf{w};\mathcal{B}_s)} \sum_{\mathbf{k}\in\mathbb{N}_0^s\setminus\{\mathbf{0}\}} \rho(\gamma;\mathbf{w};\mathcal{B}_s;\mathbf{k}) \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x}_n) \right|^2 \right)^{\frac{1}{2}},$$

where the coefficient $\rho(\gamma;\mathbf{w};\mathcal{B}_s;\mathbf{k})$ and the constant $C(\gamma;\mathbf{w};\mathcal{B}_s)$ are defined respectively by the equalities (3.71) and (3.72).

In the next theorem, we show that the anchored weighted $(\Gamma_{\mathcal{B}_s};\gamma;\mathbf{w})-$diaphony is a quantitative measure for the irregularity of the distribution of sequences. The following theorem holds:

Theorem 3.6 *The sequence ξ is uniformly distributed in $[0,1)^s$ if and only if the limit equality*

$$\lim_{N\to\infty} F_N(\Gamma_{\mathcal{B}_s};\gamma;\mathbf{w};\xi) = 0$$

holds for each choice of the vector γ of coordinate weights and the anchor vector $\mathbf{w}$.

3.12 A relationship between the mean square worst-case error and the anchored weighted $(\Gamma_{\mathcal{B}_s};\gamma;\mathbf{w})-$diaphony

In this paragraph of the thesis, we show the relationship that exists between the mean square worst-case error of the integration in the space $H_{Sob,s,\gamma,\mathbf{w}}$ and the anchored weighted $(\Gamma_{\mathcal{B}_s};\gamma;\mathbf{w})-$diaphony of the net of the nodes of the integration.

Theorem 3.7 *Let P_N be an arbitrary net of N points in $[0,1)^s$. Then, the mean square worst-case error $\tilde{e}(H_{Sob,s,\gamma,\mathbf{w}};P_N)$ of the integration in the space $H_{Sob,s,\gamma;\mathbf{w}}$, by using the net P_N of nodes, and the anchored weighted diaphony $F(\Gamma_{\mathcal{B}_s};\gamma;\mathbf{w};P_N)$ of this net are related to the equality*

$$\tilde{e}(H_{Sob,s,\gamma,\mathbf{w}};P_N)=\sqrt{C(\gamma;\mathbf{w};\mathcal{B}_s)}.F(\Gamma_{\mathcal{B}_s};\gamma;\mathbf{w};P_N),$$

where the constant $C(\gamma;\mathbf{w};\mathcal{B}_s)$ is defined by the equality (3.72).

Contributions of the author

According to the presented in the thesis results, the author has the claim to following contributions:

1. Define of the notion of multidimensional modified integrals of the functions of the system $\Gamma_{\mathcal{B}_s}$ and prove some of their properties. Some classical results of the quantitative theory of the uniformly distributed sequences, as the LeVeque's inequality, the Koksma's formula and the Erdös-Turàn-Koksma inequality are presented in the terms of the introduced modified integrals;

2. The integral Weyl's criterion that a sequence to be uniformly distributed is presented in the terms of these integrals;

3. A definition of the notion of mean square worst-case error of the integration in Hilbert spaces, which is based on the arithmetic related to the function system $\Gamma_{\mathcal{B}_s}$ is introduced. The presentation of this mean square worst-case error as an ordinary worst-case error of the integration in Hilbert spaces, generated by the associated digitally $\mathcal{B}_s-$adic shifted kernel is proved;

4. Two types of Sobolev spaces $-$ the unanchored weighted Sobolev space $H_{Sob,s,\gamma}$ and the anchored weighted Sobolev space $H_{Sob,s,\gamma,\mathbf{w}}$ are

considered. In the both cases of spaces, the formulas in explicit form for the mean square worst-case error of the integration in these spaces are proved.

5. The notions of the unanchored and anchored weighted diaphony are introduced and it is proved that these types of the diaphony are quantitative measures for the irregularity of the distribution of sequences in $[0,1)^s$.

6. The relationships that exist between the mean square worst-case errors of the integration in the spaces $H_{Sob,s,\gamma}$ and $H_{Sob,s,\gamma,\mathbf{w}}$ and the corresponding types of the diaphony - the unanchored weighted diaphony $(\Gamma_{\mathcal{B}_s};\gamma)-$diaphony and the anchored weighted diaphony $(\Gamma_{\mathcal{B}_s};\gamma;\mathbf{w})-$diaphony are proved. In this way, the nature of the introduced types of the diaphony is explained $-$ they are the mean square worst-case errors of the integration in appropriate Sobolev spaces.

A list of the publications of the author connected with the thesis

1. V. Grozdanov and **E. Shabani**, *Multidimensional quasi-Monte Carlo integration in weighted anchored Sobolev spaces*, Comp. Rendus Akad. Bulgare Sci. **77** (12) (2024), 1743-1751.

2. V. Grozdanov and **E. Shabani**, *Some applications of the functions of the system* $\Gamma_{\mathcal{B}_s}$ *to the theory of the uniformly distributed sequences*, Annual of Sofia University "St. Kliment Ohridsi", Faculty of Mathematics and Informatics, (2025)

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[Fi 1] **N. J. Fine**, *On the Walsh functions*, J. Trans. Amer. Math. Soc. **65** (1949) 372-414.

[GrSh 1] **V. Grozdanov, E. Shabani**, *Some applications of the functions of the system $\Gamma_{\mathcal{B}_s}$ to the theory of the uniformly distributed sequences*, submit to Annual of the Sophia university St. Climent Ohridski.

[GrSh 2] **V. Grozdanov, E. Shabani**, *Multidimensional Quasi-Monte Carlo integration in weighted anchored Sobolev spaces*, accepted for publication in Comp. Rendus Akad. Bulgare Sci.

[NoWo 1] **Novak, H. Wožniakovsi**,

[Pe 1] **Ts. N. Petrova**, *Quasi-Monte Carlo integration in hybrid Korobov and Sobolev spaces*, dissertation, South West University "Neofit Rilsi", Blagoevgrad, Bulgaria, 2021.

[SlWo 1] **I. Sloan, H. Wo**ž**niakovsi**, *Tractability of multivariate integration for weighted Korobov classes*, J. Complexity **17** (2001), 697-721 doi: 10.1006/jcom.2001.0599.